On the quantum stability of the time machine

S. V. Krasnikov*

The Central Astronomical Observatory at Pulkovo, St Petersburg, 196140, Russia

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Abstract

In a number of papers it has been claimed that the time machine are quantum unstable, which manifests itself in the divergence of the vacuum expectation value of the stress-energy tensor $\langle \mathbf{T} \rangle$ near the Cauchy horizon. The expression for $\langle \mathbf{T} \rangle$ was found in these papers on the basis of some specific approach [1, 2].

We show that this approach is untenable in that the above expression firstly is not derived from some more fundamental and undeniable premises, as it is claimed, but rather postulated and secondly contains undefined terms, so that one can neither use nor check it. As a counterexample we cite a few cases of (two-dimensional) spacetimes containing time machines with $\langle \mathbf{T} \rangle$ bounded near the Cauchy horizon.

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1 Introduction

Since the wormhole-based time machine was proposed [3] much efforts have been directed towards finding a mechanism that could "protect causality" and destroy such a time machine. One of the most popular ideas is that the

*Email: *redish@pulkovo.spb.su*

creation of the time machine might be prevented by quantum effects since as it is claimed in [2] "at any event in spacetime, which can be joined to itself by a closed null geodesic, the vacuum fluctuations of a massless scalar field should produce a divergent renormalized stress-energy tensor". The considerations leading to such a claim I shall call hereafter "FKT approach".

In essence, the FKT approach amounts to the following [1, 2] (see also [4]). The vacuum expectation value of the stress-energy tensor $\langle T_{\mu\nu}\rangle$ of the field ϕ in the (multiply connected) spacetime M containing a time machine is found by applying some differential operator $D_{\mu\nu}$ to the Hadamard function

$$G^{(1)}(X,X') \equiv \langle\{\phi(X),\phi(X')\}\rangle. \tag{1}$$

To find $G^{(1)}$ it is proposed to use the formula

$$G^{(1)}(X,X') = G^{\Sigma} \equiv \sum_n \tilde{G}^{(1)}(X,\gamma^n X'). \tag{2}$$

Here $\tilde{G}^{(1)}$ is the Hadamard function of ϕ in the spacetime $\widetilde{M}$, which is the universal covering space for M, and $\gamma^n X \in \widetilde{M}$ is the n-th inverse image of $X \in M$ ($\gamma^0 X$ is identified in (2) with X). The advantage of the use of (2) is that $\tilde{G}^{(1)}$ is supposed to have the Hadamard form:

$$\tilde{G}^{(1)}(X,X') = \tilde{u}\sigma^{-1} + \tilde{v}\ln|\sigma| + \text{nonsingular terms} \tag{3}$$

where σ is half the square of the geodesic distance between X and X', and $\tilde{u}$, $\tilde{v}$ are some smooth functions. We might think thus that

$$\begin{aligned}\langle T_{\mu\nu}\rangle^{\text{ren}}_{M} &= \langle T_{\mu\nu}\rangle^{\text{ren}}_{\widetilde{M}} + \sum_{n\neq 0}\lim_{X'\to X} D_{\mu\nu}\tilde{G}^{(1)}(X,\gamma^n X') \\ &\to \langle T_{\mu\nu}\rangle^{\text{ren}}_{\widetilde{M}} + \sum_{n\neq 0}\lim_{\substack{X'\to X\\ X\to\text{horizon}}} D_{\mu\nu}(\tilde{u}\sigma_n^{-1} + \tilde{v}\ln|\sigma_n|).\end{aligned} \tag{4}$$

Here $\sigma_n \equiv \sigma(X,\gamma^n X')$ and the subscript "ren" (renormalized) has appeared because renormalization of $\langle\mathbf{T}\rangle_{\widetilde{M}}$ and of $\langle\mathbf{T}\rangle_M$ requires substraction of the same terms. The last series in (4) diverges (since $\sigma_n \to 0$, when X approaches the horizon), so the conclusion is made that the appearance of a closed timelike curve must be prevented (unless some effects of quantum gravity remedy the situation) by the infinite increase of the energy density.

The goal of this paper is to state that there is actually *no* reasons to expect that the energy density diverges at the Cauchy horizon in the general case. In particular, we cite a few examples in which the stress-energy tensor of the massless scalar field (as found in the usual way — through quantization directly in M) remains bounded as the horizon is approached. These examples are not of course anywhere near an adequate model of the time machine (for example, the wave equation on the two-dimensional cylinder has no solutions, except for constant, continuous at the Cauchy horizon). However, they can well serve as counterexamples to statements like that cited above. So it seems important to find out whether they indicate only some loop-holes in the FKT approach, which can then be used in the general case, or whether this approach must be fully revised. Our analysis in Section 2 shows that the latter is true — by several independent reasons one cannot extract any information from expression (4). In Section 3 some special cases are considered necessary to support statements from Section 2.

2 Analysis

2.1 Going to the universal covering

Formula (2), combined with some implicit assumptions serves as a basis for the overall FKT approach since one cannot use (3) in multiply connected spacetimes, where σ is not defined. To discuss its validity and to reveal these assumptions let us first state the simple fact that most properties of the Hadamard function, *including* the validity of (3), depends on the choice of the vacuum appearing in definition (1). So, formulas like (2) are meaningless until we specify the vacuua $|0\rangle$ and $|\tilde{0}\rangle$ in M and $\widetilde{M}$ respectively. We come thus to the problem of great importance in our consideration — how, given $|0\rangle$, one could determine corresponding $|\tilde{0}\rangle$? The above-mentioned assumptions concern just this problem. They must be something like the following:

1. For any vacuum on M there exists a vacuum $|\tilde{0}\rangle$ on $\widetilde{M}$ such that (2) holds.

2. The function $\widetilde{G}^{(1)}$ corresponding to $|\tilde{0}\rangle$ has the "Hadamard form" (3).

3. $G^{(1)}$ determines $\widetilde{G}^{(1)}$ uniquely.

The validity of Assumption 1 is almost obvious in the simplest cases (see below), but it was not proven in the general case. (One can meet the references to [5] in this connection. Note, however, that the functions K_C which stand there in the analog of our formula (2), are actually not defined[1] in our case, i. e. when $|\Gamma| = \infty$.)

Assumption 2 seems still more arbitrary. The validity of (3) was proven not for *any* state, but only for some specific class of states (see [6, Sect. 2c]) and there is no reason to believe that our $|\tilde{0}\rangle$ belongs just to this class.

Assumption 3 is definitely untrue. In the following section we construct as an example a class of vacuua $|\tilde{0}\rangle_f$ such that (2) is satisfied for any f while $\tilde{G}_f$ differ for different f. This nonuniqueness is far from harmless. As we argue below it makes, in fact, expression (4) meaningless.

2.2 The expression for the stress-energy tensor

Expression (4) is the main result of the FKT approach and (2) is needed only to justify it. So let us state first that

1. (4) does not follow (or, at least, does not follow immediately) from (2), since

   (a) To write $\lim D_{\mu\nu} \sum \tilde{G}^{(1)} = \sum \lim D_{\mu\nu} \tilde{G}^{(1)}$ without a special proof one must be sure that the series $\sum \tilde{G}^{(1)}$ and $\sum D_{\mu\nu} \tilde{G}^{(1)}$ converge uniformly, while it is clear that they do not (at least as long as (3) holds). This nonuniformity manifests itself, in particular, in the fact that, in general, one cannot drop the nonsingular terms in $\tilde{G}^{(1)}$. In Subsection 3.2 we shall show that the last series in (4) can diverge *off* the Cauchy horizon though (3,2) hold and $G^{(1)\mathrm{ren}}$ (and $\langle \mathbf{T} \rangle^{\mathrm{ren}}$) are smooth there.

   (b) Even when $|\tilde{0}\rangle$ belongs to the above-mentioned class, (3) is proven not for *any* X, X', but only for X' lying in the "sufficiently small" neighborhood of X. It is necessary, in particular, that $\sigma(X, X')$ would be defined uniquely. To provide this in Ref. [6], for example, X and X' are required not to lie respectively near points $\underline{x}$, and $\underline{y}$ connected by a null geodesic with a point conjugate to $\underline{x}$ before $\underline{y}$. To violate this condition for the points X' and $\gamma X'$ it suffices

[1] I am grateful to Dr. Parfyonov, who explained to me this issue.

to separate the mouths of the wormhole widely enough and to fill the space between them with the conventional matter [7, Prop. 4.4.5].

Thus, we see that (4) must be regarded as an independent assumption. We can, however, neither use nor check it in view of the aforementioned ambiguity:

2. Like the Hadamard function, $\langle T_{\mu\nu}\rangle^{\rm ren}_{\widetilde{M}}$ depends on which vacuum we choose, while from the FKT standpoint all vacuua $|\tilde{0}\rangle$ satisfying (2) are equivalent. This equivalence is of a fundamental nature — the only physical object is the spacetime M, while $\widetilde{M}$ and $|\tilde{0}\rangle$ are some auxiliary matters and as long as (2) holds we cannot apply any extraneous criteria to distinguish among them. So, we have no way of determining what to substitute in (4) as $\langle T_{\mu\nu}\rangle^{\rm ren}_{\widetilde{M}}$. In Subsection 3.2 we shall see that choosing different $|\tilde{0}\rangle$ (even when $\widetilde{M}$ is a part of the Minkowski plane) one can make $\langle T_{\mu\nu}\rangle^{\rm ren}_{\widetilde{M}}$ finite or infinite at the horizon at will.

Let me note in passing that there is no point in using (4) unless we decide that $|\tilde{0}\rangle$ is among the very "good" and convenient vacuua. For an arbitrary $|\tilde{0}\rangle$, it is not a bit easier to find $\langle \mathbf{T}\rangle^{\rm ren}_{\widetilde{M}}$ than $\langle \mathbf{T}\rangle^{\rm ren}_{M}$.

2.3 Interpretation

Suppose that $\langle\Psi|\mathbf{T}|\Psi\rangle^{\rm ren}_M$ for some $|\Psi\rangle$ does diverge at the Cauchy horizon. Suppose further that it is $\langle\mathbf{T}\rangle^{\rm ren}_M$ that stands in the right side of the Einstein equations (though it is not obvious, see [8] for the literature and discussion). Does this really mean that owing to the quantum effects the time machine M cannot be created? I think that the answer is negative. It well may be that $\langle\Phi|\mathbf{T}|\Phi\rangle^{\rm ren}_M$ does not diverge for some other state $|\Phi\rangle$ (example see in Subsection 3.3). Why must we restrict ourselves to the state $|\Psi\rangle$? To prove that the Einstein equations and QFT are incompatible in M one must have proven that the expected stress-energy tensor tends to infinity for *any* quantum state, or at least for any state satisfying some reasonable physical conditions (say, stability).

3 Examples

Let us find the expectation value of the stress-energy tensor in a few specific cases. We restrict our consideration to the two-dimensional cylinder M obtained from the plane (τ,χ) by identifying $\chi \rightsquigarrow \chi + H$ and endowed with the metric

$$ds^2 = C(-d\tau^2 + d\chi^2) = C\, dudv. \tag{5}$$

Here $u \equiv \chi - \tau$, $v \equiv \chi + \tau$; C is a smooth function on M. To find in the ordinary way $\langle \mathbf{T} \rangle$ for the free real scalar field ϕ

$$\Box\phi = 0, \qquad \phi(\chi + H, \tau) = \begin{cases} \phi(\chi,\tau) & \text{for the non-twisted field} \\ -\phi(\chi,\tau) & \text{for the twisted field} \end{cases} \tag{6}$$

we must first of all specify the vacuum we consider. That is we must choose a linear space of solutions of (6) and an "orthonormal" basis [8] $U = \{u_n\}$ in it. In particular, this will define the Hadamard function:

$$G^{(1)}(X, X') = \sum_n u_n(X) u_n^*(X') + \text{complex conjugate}.$$

A possible choice of U for the non-twisted field is

$$u_n = |4\pi n|^{-1/2} e^{2\pi i H^{-1}(n\chi - |n|\tau)} \quad n = \pm 1, \pm 2 \ldots \tag{7}$$

The vacuum $|0\rangle_C$ defined by (7) (the "conformal" vacuum) is especially attractive as the expressions for the Hadamard function $G^{(1)}_C$ and for the stress-energy tensor $\langle \mathbf{T} \rangle_C$ are already obtained (see [8, the neighborhood of formula (6.211)]):

$$\begin{aligned} \langle T_{ww} \rangle^{\text{ren}}_C &= -\frac{\pi\epsilon}{12H^2} + \frac{1}{24\pi}\left[\frac{C_{,ww}}{C} - \frac{3}{2}\frac{{C_{,w}}^2}{C^2}\right], \quad w = u, v \\ \langle T_{uv} \rangle^{\text{ren}}_C &= \langle T_{vu} \rangle^{\text{ren}}_C = -RC/(96\pi). \end{aligned} \tag{8}$$

Here $\epsilon = -1/2$ or 1 depending on whether ϕ is twisted or untwisted and R is the curvature of M. Though the absence of a solution corresponding to $n = 0$ in (7) may seem artificial, it is, in fact, an inherent feature of $|0\rangle_C$, which is to describe the vacuum of ϕ as a massless limit of the "natural" vacuum of a massive field (cf [8, below (4.220)]). One could start, however, from another vacuum for the massive field and arrive at another theory (see below) with the basis U'

$$U' = U \cup u_0 \equiv (2H)^{-1/2}(F\tau + i/F).$$

Where the real constant F is a free parameter. Choosing different $F \neq 0$ we obtain different vacuua $|0\rangle_F$ and Hadamard functions $G^{(1)}_F$. It is easy to see that

$$G^{(1)}_F = G^{(1)}_C + \frac{F^2}{H}\tau\tau' + const. \tag{9}$$

3.1 Two-dimensional time machines in the conformal vacuum state

As a first example let us consider the Misner spacetime, which is the quadrant $\alpha < 0$, $\beta > 0$ of the Minkowski plane $ds^2 = d\alpha d\beta$ with points identified by the rule $(\alpha_{0,}\beta_0) \mapsto (A\alpha_{0,}\beta_0/A)$. The coordinate transformation

$$u = -W^{-1}\ln|W\alpha|, \quad v = W^{-1}\ln(W\beta)$$

delivers the isometry between Misner space and M with

$$C = e^{2W\tau}, \qquad H = W^{-1}\ln A.$$

W here is an arbitrary parameter with dimension of mass. Substituting this in (8) we immediately find

$$\langle T_{ww'}\rangle^{\mathrm{ren}}_C = -W^2\left(\frac{\epsilon\pi}{12\ln^2 A} + \frac{1}{48\pi}\right)\delta_{ww'}.$$

The metric in coordinates α, β is "good" (smooth, nondegenerate) near the Cauchy horizons $\alpha = 0$ or $\beta = 0$. So, the proper basis of an observer approaching to one of them with a finite acceleration is related to the basis $D \equiv \{\partial_\alpha, \partial_\beta\}$ by a finite Lorentz transformation. Thus the quantities we are to examine are, in fact, the components of $\langle \mathbf{T}\rangle^{\mathrm{ren}}_C$ in the basis D, which are

$$\langle T_{\alpha\alpha}\rangle^{\mathrm{ren}}_C = T\alpha^{-2}, \quad \langle T_{\beta\beta}\rangle^{\mathrm{ren}}_C = T\beta^{-2}, \quad \langle T_{\alpha\beta}\rangle^{\mathrm{ren}}_C = \langle T_{\beta\alpha}\rangle^{\mathrm{ren}}_C = 0,$$

$$T \equiv -\left(\frac{\pi\epsilon}{12\ln^2 A} + \frac{1}{48\pi}\right).$$

Now let us use the above simple method to find $\langle \mathbf{T}\rangle^{\rm ren}$ for two time machines more (see also [9]). Consider first the cylinder S obtained from the strip

$$
\begin{gathered}
ds^2 = W^{-2}\xi^{-2}(-d\eta^2 + d\xi^2) = \xi^{-2} d\alpha d\beta, \\
\text{where } \alpha \equiv (\xi - \eta)/W,\ \beta \equiv (\xi + \eta)/W; \quad \eta \in (-\infty, \infty),\ \xi \in [1, A].
\end{gathered}
\tag{10}
$$

by gluing points $\eta = \eta_0,\ \xi = 1$ with the points $\eta = A\eta_0,\ \xi = A$. This spacetime was considered in detail in [1] where it was called the "standard model". A simple investigation shows that the Cauchy horizons $\alpha = 0$ and $\beta = 0$ divide S into three regions. Causality holds in the "inner" region $\widetilde{S}:\ \alpha, \beta > 0$ and violates in $I^{\pm}(\widetilde{S})$. Introducing new coordinates $u,\ v$:

$$
u \equiv W^{-1}\ln\alpha, \quad v \equiv W^{-1}\ln\beta
$$

we find that $\widetilde{S}$ like the Misner space[2] is isometric to M. This time

$$
C = \cosh^{-2} W\tau, \quad H = W^{-1}\ln A,
$$

which yields

$$
\begin{gathered}
\langle T_{\alpha\alpha}\rangle_C^{\rm ren} = T\alpha^{-2}, \quad \langle T_{\beta\beta}\rangle_C^{\rm ren} = T\beta^{-2}, \\
\langle T_{\alpha\beta}\rangle_C^{\rm ren} = \langle T_{\beta\alpha}\rangle_C^{\rm ren} = (1/12\pi)(\alpha+\beta)^{-2}.
\end{gathered}
\tag{11}
$$

Consider lastly the spacetime obtained by changing $\xi^{-2} \to \eta^{-2}$ in (10). This spacetime is similar to the standard model, but has a somewhat more curios causal structure — there are two causally nonconnected regions separated by the time machine. $\langle \mathbf{T}\rangle_C^{\rm ren}$ differs from that in (11) by the off-diagonal (bounded) terms

$$
\langle T_{\alpha\beta}\rangle_C^{\rm ren} = \langle T_{\beta\alpha}\rangle_C^{\rm ren} = -(1/12\pi)(\alpha-\beta)^{-2}.
$$

So, we see that in all three cases the vacuum energy density (associated with *some* vacuum states) does grow infinitely as one approaches to the Cauchy horizon. A few comments are necessary, however:

[2] In spite of their apparent similarity these spaces are significantly distinct. For example, the Misner spacetime is geodesically incomplete [7], and the standard model is not [10]. This may be of importance if one would like to separate X and X' "widely enough" (see item 1b in the previous section).

1. The divergence in discussion is not at all something peculiar to the time machine: the passage to the limit $A \to \infty$ shows that precisely the same divergence (with $T = -1/(48\pi)$) takes place in $\widetilde{M}$ though (in the case of Misner space) $\widetilde{M}$ is merely a part of the Minkowski plane. This suggests that for the time machine too, the divergence of the stress-energy tensor is a consequence not of its causal or topological structure but rather of the unfortunate choice of the quantum state.

2. The twisted field at $A = e^{\sqrt{2}\pi}$ has the bounded $\langle \mathbf{T} \rangle_C^{\rm ren}$ (cf. [11]).

3. Let us consider nonvacuum states now (see Subsection 2.3). The first example is a two-particle state $|1_n 1_{-n}\rangle$ with the particles corresponding to the n-th and $-n$-th modes of (7). $\langle 1_{-n}1_n|\mathbf{T}|1_n 1_{-n}\rangle^{\rm ren}$ is readily found using [8, eq. (2.44)]:

$$\langle 1_{-n}1_n|T_{\gamma\gamma}|1_n 1_{-n}\rangle^{\rm ren} = T'\gamma^{-2}, \quad \langle 1_{-n}1_n|T_{\alpha\beta}|1_n 1_{-n}\rangle^{\rm ren} = \langle T_{\alpha\beta}\rangle_C^{\rm ren}$$

   with $T' \equiv T + 2\pi n H^{-2}$, and $\gamma \equiv \alpha, \beta$. Thus we see that there *are* states with the bounded energy density of the untwisted field.

   Another yet example is the equilibrium state at a nonzero temperature $|t\rangle$. Expression (4.27) of [8] gives

$$\langle t|T_{ww}|t\rangle^{\rm ren} = \langle T_{ww}\rangle_C^{\rm ren} + \frac{\pi}{2H^2}\sum_{m=1}^{\infty}\sinh^{-2}\frac{\pi m}{k_B t H}.$$

   So, for any H there exists such temperature t that $\langle t|T_{\gamma\gamma}|t\rangle^{\rm ren}$ does not diverge at the horizon.

3.2 Another vacuum

The conformal vanuum is not suited for verifying or exemplifying most of statements made in Section 2., since the Hadamard function does not exist in this state. So consider now the new vacuum $|0\rangle_f$ on the plane (τ, χ) defined by the modes

$$u'_p \equiv \begin{cases} \dfrac{1}{2\sqrt{\pi\omega}} e^{ip\chi - i\omega\tau}, & \omega \geq \delta \\[2ex] \dfrac{1}{2\sqrt{\pi}} e^{ip\chi}(f^{-1}\cos\omega\tau - i\omega^{-1} f \sin\omega\tau), & \omega < \delta. \end{cases} \tag{12}$$

where $\omega \equiv |p|$, δ is an arbitrary positive constant: $\delta < 1$ and f is an arbitrary smooth positive function: $f(\omega \geq \delta) = \sqrt{\omega}$. The modes (12) are obtained ¿from that defining the conformal vacuum on the plane by a Bogolubov transformation of the low-frequency modes so as to avoid the infrared divergence without affecting the ultraviolet behavior of $\langle \mathbf{T} \rangle$. The asymptotic form of $\tilde{G}^{(1)}_f$ does not depend on f:

$$\forall f \qquad \tilde{G}^{(1)}_f = -1/(2\pi) \ln |\Delta u \Delta v| + \text{smooth, bounded function.} \tag{13}$$

If we retain only the first term, we obtain (in the flat case)

$$\lim_{X' \to X} D_{\alpha\alpha} \tilde{G}^{(1)}(X, \gamma^n X') = \ln^2(A)\, \alpha^{-2} A^{-n} n^{-2}.$$

So, the last series in (4) diverges not only at the horizon, but everywhere on M (cf. Subsection 2.2).

$\langle \mathbf{T} \rangle_f$ can be found from (9) (see [8, Sect. 6.4]). For any C we have:

$$\langle T_{ww} \rangle^{\text{ren}}_f = \frac{1}{8\pi} \left[-\frac{W^2}{6} + \int_0^\delta (f^{-2}\omega^2 + f^2 - \omega)\, d\omega \right], \quad \langle T_{uv} \rangle^{\text{ren}}_f = \langle T_{uv} \rangle^{\text{ren}}_C.$$

Having taken an appropriate $f(\omega)$ one can make $\langle T_{\alpha\alpha} \rangle^{\text{ren}}_f$ infinite or zero at the horizon, as we have stated in Subsection 2.2.

To illustrate some more statements from Section 2. let us first find G^Σ. To this end note that it has the form

$$G^\Sigma = \sum_n \int_{-\infty}^{\infty} h(p) e^{inHp}\, dp + c.\ c. \tag{14}$$

with

$$h \equiv \begin{cases} \dfrac{1}{4\pi\omega} e^{ip\Delta\chi - i\omega\Delta\tau}, & \omega \geq 1/2 \\[2ex] \dfrac{1}{4\pi} e^{ip\Delta\chi} (f^{-2} \cos\omega\tau \cos\omega\tau' + \omega^{-2} f^2 \sin\omega\tau \sin\omega\tau'), & \omega < 1/2. \end{cases}$$

The function $h(p)$ can be written as a sum: $h = (h - h_0) + h_0$, where

$$h_0 \equiv \frac{1}{4\pi\sqrt{1+p^2}}\, e^{ip\Delta\chi}\, e^{-i\sqrt{1+p^2}\Delta\tau}.$$

The first summand is a smooth function falling off at infinity like p^{-2} and the second summand (h_0) is a holomorphic (but for $p = \pm i$) function admitting the following estimate:

$$|h_0| \leq C|x|^{-1/2}\, e^{|(\Delta\chi - \Delta\tau)y|}.$$

Hence [12] we can apply the Poisson formula to (14) and obtain:

$$G^{\Sigma} = 2\pi H^{-1} \sum_n h(2\pi H^{-1} n) + c.\ c.$$

We see thus that G^{Σ} is indeed the Hadamard function and it corresponds to the vacuum $|0\rangle_F$ with $F = f(0)$.

Remark 1. This does not mean, however, that G^{Σ} will be a Hadamard function of some reasonable state for *any* $\widetilde{G}^{(1)}$. One can easily construct, for example, such a vacuum that $\widetilde{G}^{(1)}(\chi, \chi')$ will not be invariant under translations $\chi, \chi' \mapsto \chi + H, \chi' + H$ and $G^{\Sigma}(\chi, \chi')$, as a consequence, will not even be symmetric.

Remark 2. For all $G_f^{(1)}$ with the same $f(0)$ the Hadamard functions G^{Σ} are the same. This proves our statement from Subsection 2.1.

To find $\langle \mathbf{T} \rangle_F$ note that it differs from $\langle \mathbf{T} \rangle_C$ only by the term arising from the second summand in (9) (cf. [8, eqs. (4.20), (6.136)]):

$$\Delta \langle T_{ww'} \rangle^{\mathrm{ren}} = \frac{F^2}{2H} (\tau_{,w}\, \tau_{,w'} - 1/2\, \eta_{ww'} \eta^{\lambda\delta} \tau_{,\lambda}\, \tau_{,\delta}\,).$$

That is

$$\begin{aligned} \langle T_{ww} \rangle_F^{\mathrm{ren}} &= \frac{F^2}{8H} - W^2 \left(\frac{\epsilon\pi}{12 \ln^2 A} + \frac{1}{48\pi} \right), \\ \langle T_{uv} \rangle_F^{\mathrm{ren}} &= \langle T_{vu} \rangle_F^{\mathrm{ren}} = \langle T_{uv} \rangle_C^{\mathrm{ren}}. \end{aligned}$$

So, for all three time machines considered here there exists a vacuum, such that the expectation value of the stress-energy tensor is bounded in the causal region.

4 Conclusion

Thus, we have seen that one cannot obtain any information about the energy density near the Cauchy horizon employing the FKT approach. In the absence of any other general approach this means that all we have is a few simple examples. In some of them the energy density diverges there and in some do not. So, the time machine perhaps is stable and perhaps is not. This seems to be the most strong assertion we can make.

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